Teo: $X_1, \dots, X_k \in \mathfrak{X}(M)$ $\quad p \in M$

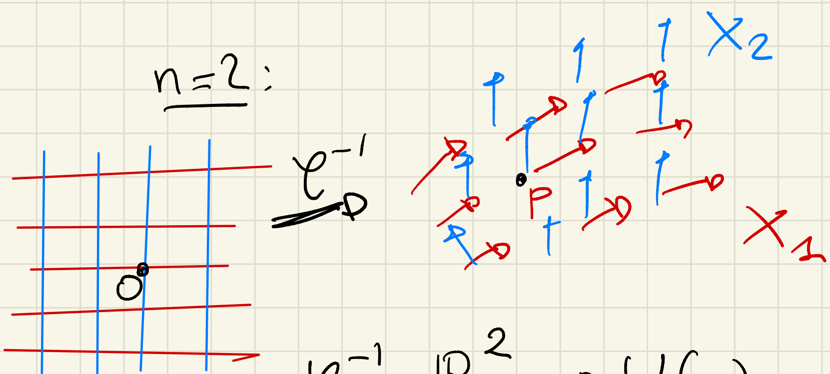
$\exists\ U(p) \xrightarrow[\varphi]{\sim} \mathbb{R}^n$ carta che porta X_i in $\frac{\partial}{\partial x^i}$

$\iff$ 1) $X_1(p), \dots, X_k(p)$ indipendenti

2) $[X_i, X_j] \equiv 0$ in un intorno di p

$\boxed{\Leftarrow}$ caso $k=n$ $\quad M = \mathbb{R}^n$

<u>$n=2$:</u>



$[X_1, X_2] \equiv 0$

$\Downarrow$

I flussi commutano

$\mathbb{R}^n$

$\varphi^{-1}: \mathbb{R}^2 \to U(p)$

$(t, u) \longmapsto$ punto di arrivo dopo t secondi lungo X_1

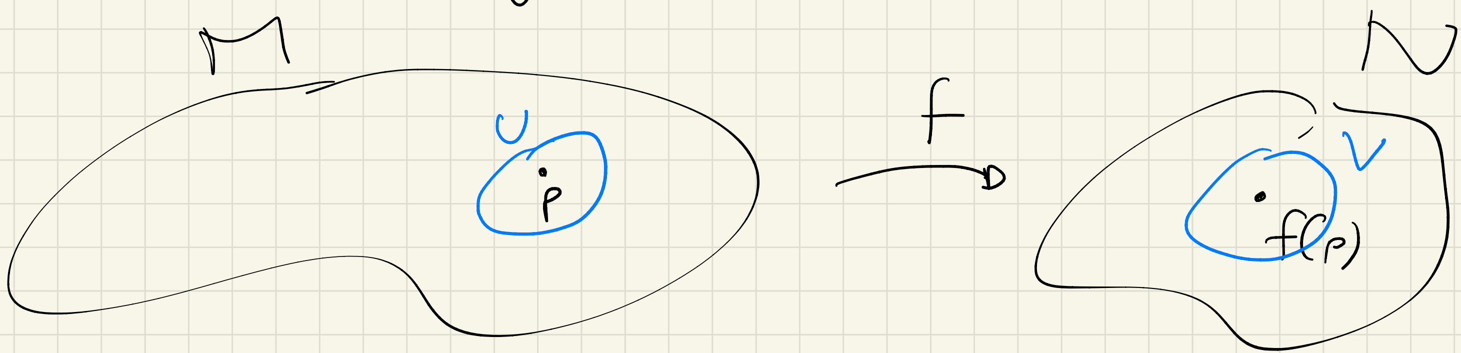
e u secondi lungo X_2

$$F_2(F_1(p,t),u) = F_1(F_2(p,u),t)$$

DIFFEOM. LOCALE

Def: $f: M^n \to N^n$ è diffeom. locale in $p \in M$ se $\exists\ U(p),\ V(f(p))$ t.c.

1) $f(U) = V$
2) $f|_U : U \to V$ diffeo

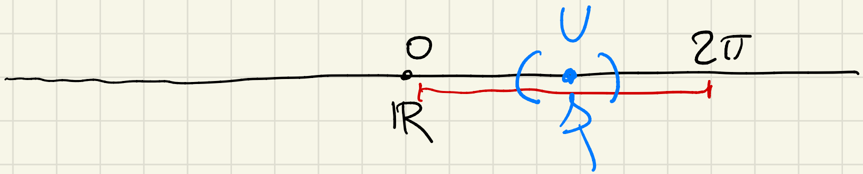


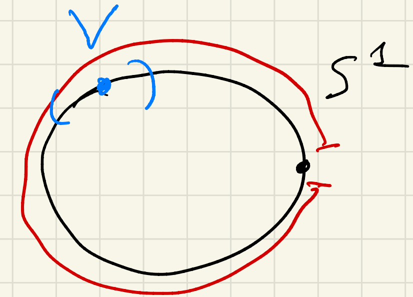
Def: f è DIFFEO LOC se lo è $\forall p \in M$

Es: $f: \mathbb{R} \to S^1$ (rivestimento)

$t \mapsto e^{2\pi i t}$

non è iniettiva

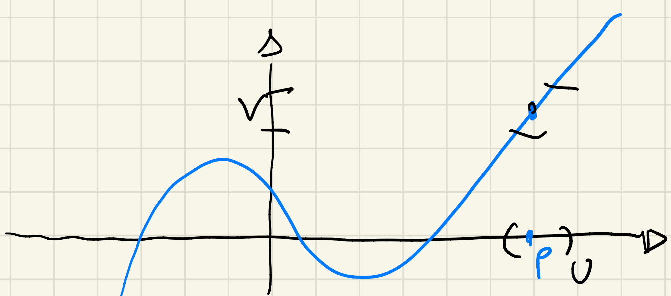




Teo (INVERTIBILITÀ LOCALE)

$\varphi: M^n \to N^n$ $\quad p \in M$

φ è diffeo loc. in p $\iff$ $d\varphi_p : T_pM \to T_{\varphi(p)}N$ è isomorfismo

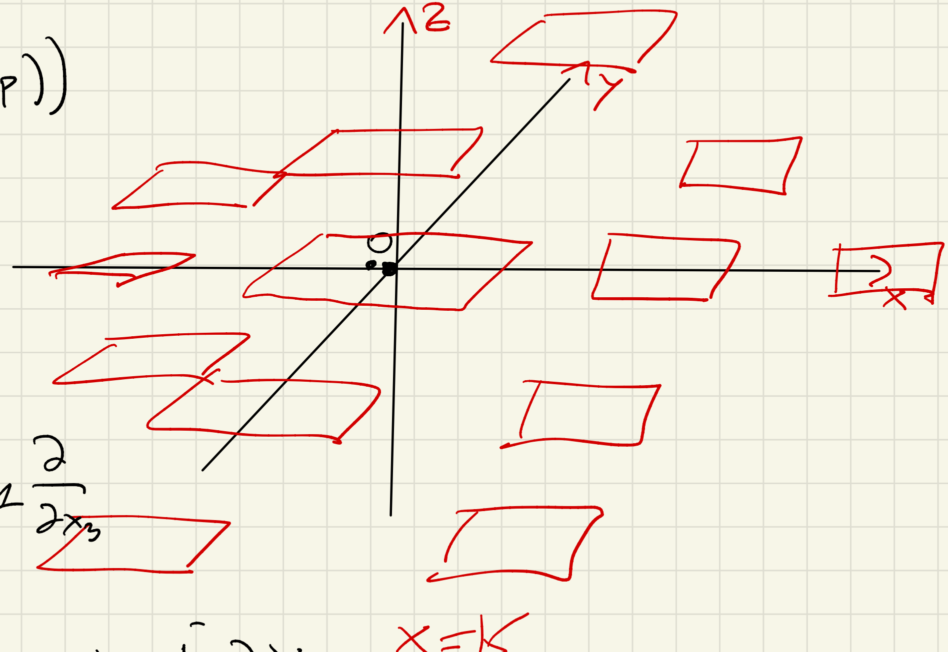


Applicazione del Teo Frobenius : $\mathbb{R}^3$

$$X_1 = \frac{\partial}{\partial x} \qquad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \qquad \text{indip. } \forall p = (x, y, z)$$

$$D(p) = \operatorname{Span}\left(X_1(p), X_2(p)\right)$$

non è integrabile



$$X_1 = \frac{\partial}{\partial x_1} \qquad X_2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}$$

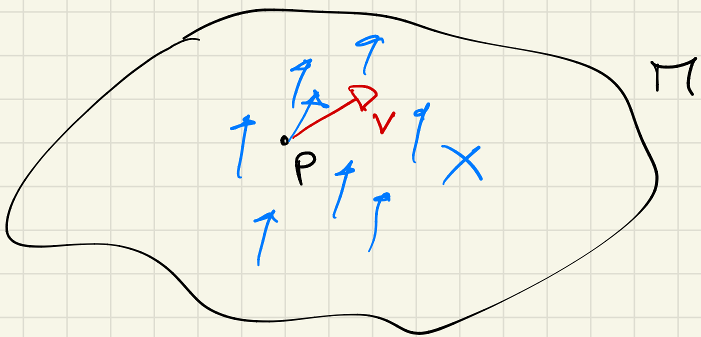
$$[X_1, X_2] = X_1^i \frac{\partial X_2}{\partial x^i} - X_2^i \frac{\partial X_1}{\partial x^i}$$

$$= \frac{\partial}{\partial x_3} = \boxed{\frac{\partial}{\partial z}} \neq 0$$

DERIVATA DI LIE

D: Se $p \in M$ $v \in T_pM$ $\quad f : M \to \mathbb{R}$

La "derivata" di f rispetto a v è definita $v(f) \in \mathbb{R}$

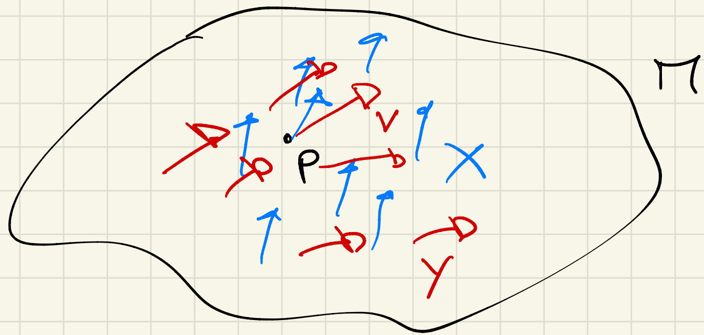


$X \in \mathfrak{X}(M)$

NON ESISTE una nozione naturale di derivata di X rispetto a v.

Oss: Se $X(p) \neq 0$

$\exists$ carte in cui $X = \frac{\partial}{\partial x_1}$



$[X,Y] = -[Y,X]$

identità di Jacobi

$[X,[Y,Z]] +$
$[Y,[Z,X]] +$
$[Z,[X,Y]] = 0$

$\mathcal{L}$ DERIVATA DI LIE

Potrei definire $\mathcal{L}_Y(X) = [Y,X]$

In realtà $\mathcal{L}$ si definisce in modo più generale così:

Sia S un campo tensoriale in M di tipo (h,k)

$X \in \mathfrak{X}(M)$

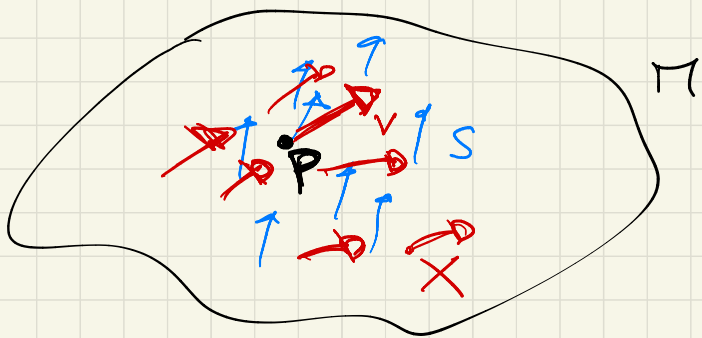
Def: $\mathcal{L}_X(S)$ DERIVATA DI LIE

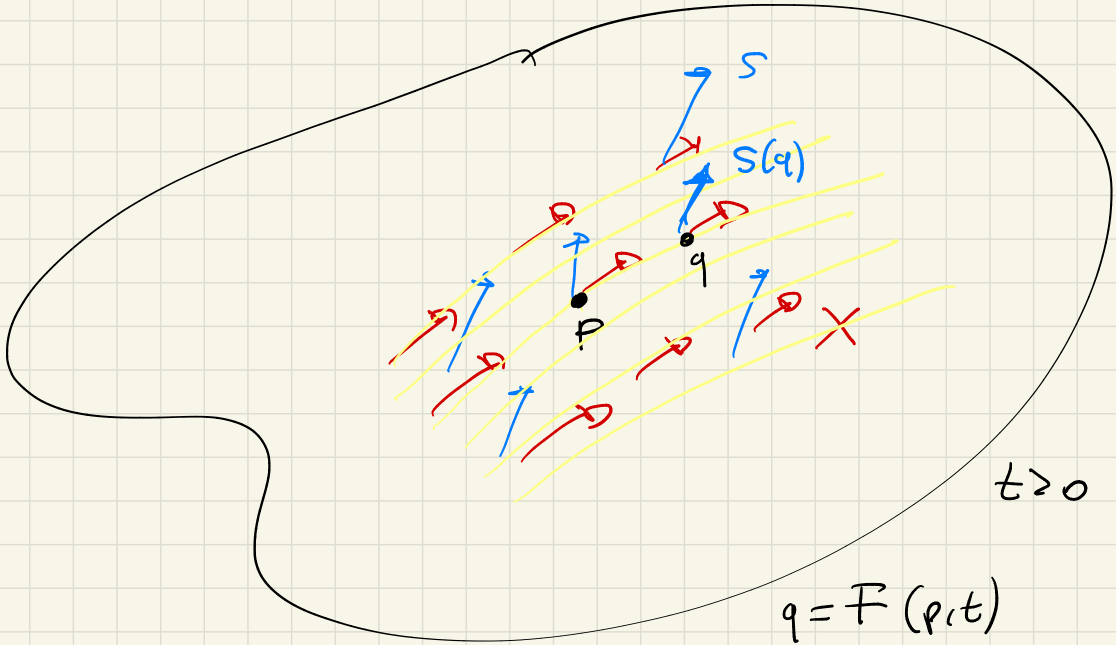
è un nuovo campo tensoriale (h,k)

X ----→ flusso $F : (M \times \mathbb{R}) \to M$

$\cup$

U





$$q = F(p,t)$$

$$(dF_t)_p : T_pM \xrightarrow{\sim} T_qM$$

$$v_t \longleftarrow s(q)$$

$$\mathcal{L}_X(s)(p) = \frac{dv_t}{dt}\Big|_{t=0}$$

Proprietà: 1) Se s è funzione $\mathcal{L}_X(s) = X(s)$

2) Se $s \in \mathcal{X}(M)$

$$\mathcal{L}_X(s) = [X, s]$$

3) $\mathcal{L}_X(s \otimes s') = \mathcal{L}_X(s) \otimes s' + s \otimes \mathcal{L}_X(s')$